THERMAL BOUNDARY LAYER ON A PLATE IN A NON-NEWTONIAN FLUID WITH NONLINEAR HEAT CONDUCTION LAW

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The flat-plate boundary layer equation for a rheological power law and a proposed nonlinear law of heat conduction is reduced to an ordinary differential equation, which is solved in quadratures using previously calculated [2] velocity profiles. Graphs of the temperature and heat transfer coefficient profiles are presented.

We write the equation of the thermal boundary layer neglecting viscous dissipation:

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=-\frac{\partial q_y}{\partial y}. \qquad (1)$$

In order to solve this equation it is necessary to have the velocity profiles u(x, y) and v(x, y), obtained by solving the dynamic equations, and to specify the law of heat conduction.

For a fluid governed by the rheological power law [1]

$$\bar{\bar{\tau}}=K\,(2I_2)^{\frac{n-1}{2}}\bar{\bar{e}}, \qquad (2)$$

in the case of a plate and a Fourier law of heat conduction at $n\neq 1$ Eq. (1) does not have similar solutions.

Two possible forms of the law of heat conduction ensuring self-similarity of (1) in the case of a plate and fluid (2) are

$$\mathbf{q}=-H\,(2I_2)^{\frac{n-1}{2}}\,\mathrm{grad}\;T, \qquad (3)$$

$$\mathbf{q}=-N\,|\,\mathrm{grad}\,T\,|^{n-1}\,\mathrm{grad}\,T. \qquad (4)$$

Law (3) can be justified on the basis of phenomenological considerations; law (4) for plane flow is analogous in form to law (2). At n = 1 both laws go over into the Fourier law.

We will derive a solution of Eq. (1) for law (4).

Using Eq. (4), we reduce Eq. (1) to the dimensionless form

$$u_1\frac{\partial\theta}{\partial x_1}+v_1\frac{\partial\theta}{\partial y_1}=-\frac{1}{s}\frac{\partial}{\partial y_1}\left(-\frac{\partial\theta}{\partial y_1}\right)^n.^{*} \qquad (5)$$

Here,

$$u_1=\frac{u}{U},\quad v_1=\frac{v}{U}\,\mathrm{R}^{\frac{1}{1+n}},\quad x_1=\frac{x}{L},$$

$$y_1=\frac{y_1}{L}\,\mathrm{R}^{\frac{1}{1+n}},\qquad \theta=\frac{T-T_\infty}{T_w-T_\infty},$$

$$s=\frac{Kc_p}{N}\left(\frac{U}{T_w-T_\infty}\right)^{n-1},\ \mathrm{R}=\frac{\rho U^{2-n}L^n}{K}. \qquad (6)$$

The boundary conditions of Eq. (5) have the form

$$\theta=1\ \text{ at }\ y_1=0;\ \theta=0\ \text{ at }\ y_1=\infty. \qquad (7)$$

At s = 1 Eq. (5) for θ is identical with the equation of the dynamic boundary layer for u_1 [2]; in this case in accordance with boundary conditions (7), as for law (3) when $\sigma = Kc_p/H = 1$, we have the similarity relation

$$\theta=1-u_1\quad(s=1). \qquad (8)$$

As in [2], going over in (5) from x_1, y_1 to the Prandtl-Mises variables x_1, $\psi_1(x_1, y_1)$, we obtain

$$\frac{\partial\Phi}{\partial x_1}=\frac{u_1}{s}\frac{\partial}{\partial\psi_1}\left(\frac{\partial\Phi}{\partial\psi_1}\right)^n, \qquad (9)$$

where

$$d\Phi=-u_1\,d\theta. \qquad (10)$$

By means of the substitution

$$\zeta=\psi_1\left[\sqrt{2}\,n(1+n)x_1\right]^{-\frac{1}{1+n}} \qquad (11)$$

we transform (9) into the ordinary equation

$$-\zeta=\frac{\sqrt{z}}{s}(\Phi')^{n-2}\,\Phi''=\frac{\sqrt{z}}{s(n-1)}\frac{d}{d\zeta}(\Phi')^{n-1}. \qquad (12)$$

Here, the prime denotes the derivative with respect to ζ, and the function

$$z=u_1^2/2 \qquad (13)$$

satisfies the dynamic equation [2]

$$-\zeta=\sqrt{z}\,(z')^{n-2}z''=\frac{\sqrt{z}}{n-1}\frac{d}{d\zeta}(z')^{n-1}, \qquad (14)$$

whose approximate solution has the form [2]

$$z=C_0\int_0^\zeta\exp(-\zeta^2)\,d\zeta\qquad(n=1),$$

$$z=\int_0^\zeta\left[(n-1)(C-\zeta^2)\right]^{\frac{1}{n-1}}d\zeta\quad(n\neq 1). \qquad (15)$$

After transformation, boundary conditions (7) take the form

$$\theta=1\ \text{ at }\ \zeta=0,$$
$$\theta=0\ \text{ at }\ \zeta=\infty. \qquad (16)$$

From (10) and (13), using the first of conditions (16), it follows that

$$\theta(\zeta)=1-\int_0^\zeta\frac{\Phi'\,d\zeta}{\sqrt{2z}}. \qquad (17)$$

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*In the given problem $\partial\theta/\partial y_1 < 0$.

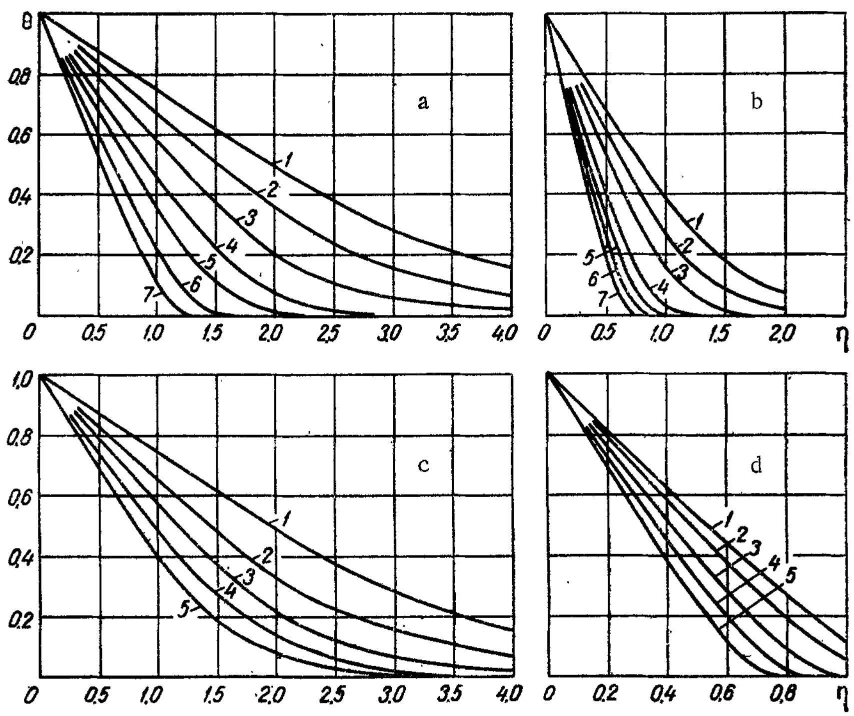

Fig. 1. Temperature profiles in the boundary layer: a and b) at s = 1 and 20 (1, at n = 0.33; 2, 0.5; 3, 0.71; 4, 1.0; 5, 1.33; 6, 2.0; 7, 3.0); c and d) at n ≅ 0.33 and n = 3 (1, at s = 1; 2, 2; 3, 5; 4, 10; 5, 20).

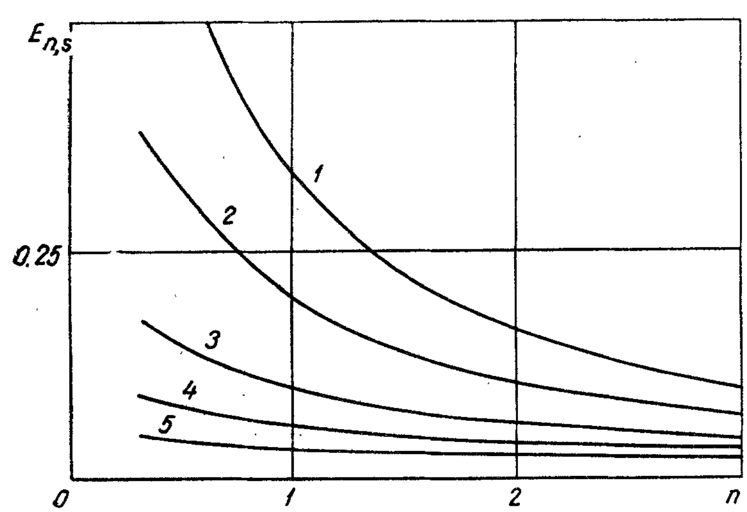

Fig. 2. Heat transfer characteristics: 1) at s = 1; 2) 2; 3) 5; 4) 10; 5) 20.

The integration of Eq. (12) using (14) and (15) gives

$$\Phi' = C_1 \exp(-s\zeta^2) \quad (n = 1),$$

$$\Phi' = [s(n-1)(C_2 - \zeta^2)]^{\frac{1}{n-1}} \quad (n \neq 1), \tag{18}$$

where C_1 and C_2 are arbitrary constants. Then, in accordance with (17), (18), and (15),

$$\theta(\zeta) = 1 - C_3 \int_0^{\zeta} \frac{\exp(-s\xi^2)\, d\xi}{\sqrt{\int_0^{\xi} \exp(-\nu^2)\, d\nu}} \quad (n = 1), \tag{19}$$

$$\theta(\zeta) = 1 - \int_0^{\zeta} \frac{[s(n-1)(C_2 - \xi^2)]^{\frac{1}{n-1}}}{\sqrt{\int_0^{\xi} [(n-1)(C - \nu^2)]\, d\nu}} d\xi \quad (n \neq 1). \tag{20}$$

The constant C in (20) is known from the solution obtained in [2]; for $n \le 1$ C_3 and C_2 are determined from the second of conditions (16), for $n > 1$ from the condition

$$\theta = 0, \ \theta' = 0 \ \text{ at } \ \zeta = \zeta_\delta, \tag{21}$$

where ζ_δ corresponds to the finite thickness of the boundary layer [2]. From (20) it follows that $\zeta_\delta = (C_2)^{1/2}$.

The constant C_2 was determined by the method of chords [3].

Figure 1 presents examples of the calculation of profiles of θ as a function of η, where

$$\eta = y_1 [\sqrt{2}\, n(1+n) x_1]^{-\frac{1}{1+n}} = \int_0^{\zeta} \frac{d\zeta}{u_1(\zeta)}. \tag{22}$$

It is clear that as s increases, the thickness of the thermal boundary layer decreases both for $n \le 1$ and for $n > 1$.

Using Eqs. (4), (6), (10), and (22), we determine the local heat transfer coefficient

$$\mathrm{St} = \frac{q_y|_{y=0}}{\rho U c_p (T_w - T_\infty)} = \frac{E_{ns}}{R_x^{\frac{1}{1+n}}}, \tag{23}$$

where

$$E_{ns} = \frac{[\sqrt{2}\, n(1+n)]^{-\frac{n}{1+n}} [\Phi'(0)]^n}{s}, \quad R_x = \frac{\rho U^{2-n} x^n}{K}. \tag{24}$$

Values of E_{ns} are presented in Fig. 2.

NOTATION

ρ is the density of fluid; c_p is the specific heat; x is the longitudinal coordinate; y is the transverse coordinate; u and v are the velocity vector components along the x- and y-axes, respectively; T is the absolute temperature; T_w is the same at the wall; T_∞ is the same in the external flow; q is the conductive heat flux vector; $\bar{\bar{\tau}}$ is the viscous stress tensor; $\bar{\bar{e}}$ is the strain rate tensor; I_2 is the second invariant of tensor $\bar{\bar{e}}$; K and n are the rheological characteristics of the fluid; H and N are the heat conduction characteristics; U is the external flow velocity; L is the characteristic length; R is the Reynolds number; R_x is the local Reynolds number; s and σ are the generalized Prandtl numbers; St is the Stanton number.

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